10/29 <u>Triple Integrals</u>

<u>IDEA:</u> Integrate functions of 3 variables

Remark: all the hard work to up the dimension was already done

• 1-variable $\to$ 2-variable was the hardest part

Conceptually, this is no different from double integrals, but pictures are hard.

<u>Main Technique:</u>

$\iiint_R f(x,y,z)\,dV$ is computable via an iterated integral

- Same as before, order of integration is more-or-less up to us as long as we parameterize appropriately

Ex: compute $\iiint_E (xy+z^2)\,dV$ for $E=[0,2]\times[0,1]\times[0,3]$
(Rectangular Prism)

$$= \left\{ (x,y,z) : \begin{array}{c} 0\le x\le 2 \\ 0\le y\le 1 \\ 0\le z\le 3 \end{array} \right\}$$

Sol.: $$\int_{x=0}^{2}\int_{y=0}^{1}\int_{z=0}^{3} (xy+z^2)\,dz\,dy\,dx$$

<u>Innermost:</u> $$\int_{z=0}^{3} (xy+z^2)\,dz = \left[xyz+\tfrac{1}{3}z^3\right]_{z=0}^{3} = (3xy+9)-0$$

<u>Middle:</u> $$\int_{y=0}^{1} (3xy+9)\,dy = \left[\tfrac{3}{2}xy^2+9y\right]_{y=0}^{1} = \left(\tfrac{3}{2}x+9\right)-0$$

<u>Outermost:</u> $$\int_{x=0}^{2} \left(\tfrac{3}{2}x+9\right)dx = \left[\tfrac{3}{4}x^2+9x\right]_{x=0}^{2} = \left(\tfrac{3}{4}\cdot 4+18\right)-0 = \boxed{21}$$

Ex: Compute $\iiint_R (2x-y)\,dV$ where $R = \{(x,y,z) : 0 \le z \le 2,\ 0 \le y \le z^2,\ 0 \le x \le y-z\}$

Note: this parameterization has the form

$\{(x,y,z) : c_1 \le z \le c_2,\ g_1(z) \le y \le g_2(z),\ h_1(y,z) \le x \le h_2(y,z)\}$

This has the same form as what we liked when we computed double integrals

Sol.

$$= \int_{z=0}^{2} \int_{y=0}^{z^2} \int_{x=0}^{y-z} 2(x-y)\,dx\,dy\,dz$$

Innermost:

$$\int_{x=0}^{y-z} (2x-y)\,dx = \Big[x^2 - xy\Big]_{x=0}^{y-z} = (y-z)^2 - (y-z)y - 0$$
$$= y^2 - 2yz + z^2 - y^2 + yz = z^2 - yz$$

Middle:

$$\int_{y=0}^{z^2} (z^2 - yz)\,dy = \Big[yz^2 - \tfrac{1}{2}y^2 z\Big]_{y=0}^{z^2} = z^2 \cdot z^2 - \tfrac{1}{2}(z^2)^2 z - 0 = z^4 - \tfrac{1}{2}z^5$$

Outermost:

$$\int_{z=0}^{2} \left(z^4 - \tfrac{1}{2}z^5\right) dz = \Big[\tfrac{1}{5}z^5 - \tfrac{1}{12}z^6\Big]_{z=0}^{2} = \frac{32}{5} - \frac{64}{12} - 0 = \frac{32}{5} - \frac{16}{3}$$
$$= \boxed{\frac{16}{15}}$$

Remark on Reparameterization:

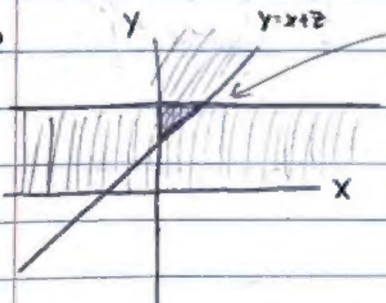
To change the order of integration, we must reparameterize to look something like this.

For this region R in the previous example, to change the order to $dy\,dx\,dz$, we need a reparameterization of the form

$R = \{(x,y,z) : c_1 \le z \le c_2,\ g_1(z) \le x \le g_2(z),\ h_1(x,z) \le y \le h_2(x,z)\}$

Look at $z = z_0$ cross-section - effectively fixes z as a constant

$z = z_0$



$z^2 = y$

$x = y - z$

$y = x + z$

$z^2 = y = x + z$

$x = z^2 - z$

$$\therefore R = \left\{ (x,y,z) :\ \begin{array}{l} 0 \le z \le 2 \\ 0 \le x \le z^2 - z \\ x + z \le y \le z^2 \end{array} \right.$$

$0 \le x \le z^2 - z$

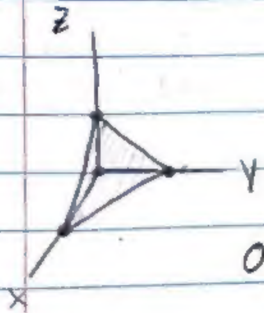
$x + z \le y \le z^2$

Note: will be fully worked in a PDF on the website

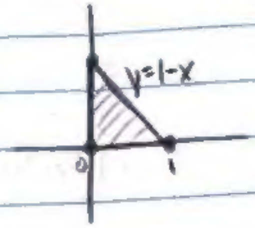
Ex.: Compute the volume of the tetrahedron T with vertices

$(0,0,0)$ $(1,0,0)$ $(0,1,0)$ $(0,0,1)$

Picture:



xy-shadow



$0 \le x \le 1$

$0 \le y \le 1-x$

$0 \le z \le 1-x-y$

$\vec{u} = \langle 1-0,\ 0-0,\ 0-1 \rangle = \langle 1, 0, -1 \rangle$

$\vec{v} = \langle 0-0,\ 1-0,\ 0-1 \rangle = \langle 0, 1, -1 \rangle$

$\vec{n} = \vec{u} \times \vec{v} = \langle 1, 1, 1 \rangle$

$\therefore$ the plane has formula $\vec{n} \cdot (\vec{x} - \vec{p}) = 0$

i.e. $\langle 1,1,1 \rangle \cdot \langle x-0,\ y-0,\ z-1 \rangle = x+y+z-1 = 0$

$z = 1-x-y$

$\therefore T = \{(x,y,z) : 0 \le x \le 1,\ 0 \le y \le 1-x,\ 0 \le z \le 1-x-y\}$

$$\therefore \mathrm{Vol}(T) = \iiint 1\, dV = \int_{x=0}^{1} \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} 1\, dz\, dy\, dx$$

$$= \int_{x=0}^{1} \int_{y=0}^{1-x} \Big[z \Big]_{z=0}^{1-x-y} dy\, dx = \int_{x=0}^{1} \int_{y=0}^{1-x} (1-x-y-0)\, dy\, dx$$

$$\int_{x=0}^{1} \Big[y - xy - \tfrac{1}{2}y^2 \Big]_{y=0}^{1-x} dx = \int_{x=0}^{1} \Big((1-x) - x(1-x) - \tfrac{1}{2}(1-x)^2 \Big) dx$$

$$= \int_{x=0}^{1} \Big((1-x)^2 - \tfrac{1}{2}(1-x)^2 \Big) dx = \frac{1}{2} \int_{x=0}^{1} (1-x)^2\, dx = \frac{1}{2} \cdot -\frac{1}{3} \Big[(1-x)^3 \Big]_{x=0}^{1}$$

$$= -\frac{1}{6}(0-1) = \boxed{\frac{1}{6}}$$